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LETTER TO THE EDITOR

Higher-dimensional integrable systems from multilinear evolution equations

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Abstract. A multilinear M-dimensional generalization of Lax pairs is introduced and its explicit form is given for the recently discovered class of time-harmonic, integrable, hypersurface motions in $\mathbb{R}^{M+1}$.

In [1] the explicit form of a triple (L, M_1, M_2), depending on two spectral parameters and four time-dependent functions $x_i(t, \varphi^1, \varphi^2, \varphi^3)$ from a three-dimensional Riemannian manifold Σ to $\mathbb{R}$ was given such that (with ρ a non-dynamical density on Σ)

$$\dot{L} = \frac{1}{\rho}\varepsilon_{rsu}\frac{\partial L}{\partial\varphi^u}\frac{\partial M_1}{\partial\varphi^r}\frac{\partial M_2}{\partial\varphi^s} \tag{1}$$

is equivalent to the equations

$$\dot{x}_i = \frac{1}{\rho}\frac{\varepsilon_{ii_1i_2i_3}\varepsilon_{r_1r_2r_3}}{3!}\partial_{r_1}x_{i_1}\partial_{r_2}x_{i_2}\partial_{r_3}x_{i_3} \tag{2}$$

describing the integrable motion of a hypersurface $\hat{\Sigma}$ in $\mathbb{R}^4$ whose time-function (the time at which $\hat{\Sigma}$ reaches a point $\boldsymbol{x} \in \mathbb{R}^4$) is harmonic [2].

The purpose of this letter is to give the explicit generalization of this construction to an arbitrary number of dimensions, $M (= \dim\Sigma)$. Let

$$z_1 = x_1 + \mathrm{i}x_2 \qquad z_2 = x_3 + \mathrm{i}x_4 \quad \ldots\ . \tag{3}$$

For *even* M $(= 2m)$ one may take

$$\begin{aligned} L &= \sum_{a=1}^{m}\left(\lambda_a z_a - \frac{\bar{z}_a}{\lambda_a}\right) + 2\sqrt{m}\,x_N \\ M_a &= \frac{\mathrm{i}}{2}\left(\lambda_a z_a + \frac{x_N}{\sqrt{m}}\right)\cdot 2^{1/m} \qquad a = 1, \ldots, m \\ M_{m+a'} &= \left(\frac{1}{\sqrt{m}}\right)^{1/(m-1)}\left(\frac{\bar{z}_{m+1-a'}}{\lambda_{m+1-a'}} - \frac{\bar{z}_{m-a'}}{\lambda_{m-a'}}\right) \qquad a' = 1, \ldots, m-1 \end{aligned} \tag{4}$$

depending on m spectral parameters, λ_a, and $N = M+1$ functions $x_i(t, \varphi^1, \ldots, \varphi^M)$; letting

$$\{f_1, \ldots, f_M\} := \frac{1}{\rho(\varphi^1, \ldots, \varphi^M)}\varepsilon_{r_1\ldots r_M}\partial_{r_1}f_1\ldots\partial_{r_M}f_M \tag{5}$$

$$\dot{L} = \{L, M_1, M_2, \ldots, M_{2m-1}\} \tag{6}$$

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will then be equivalent to the equations of motion (as above, $\bar{z}_a$ denoting the complex conjugate of z_a)

$$\begin{aligned}\dot{z}_a &= -\mathrm{i}\left(\frac{\mathrm{i}}{2}\right)^{m-1} \{z_a, z_{a+1}, \bar{z}_{a+1}, \ldots, z_{a-1}, \bar{z}_{a-1}, x_N\} \\ \dot{x}_N &= \left(\frac{\mathrm{i}}{2}\right)^{m} \{z_1, \bar{z}_1, \ldots, z_m, \bar{z}_m\}\,.\end{aligned} \tag{7}$$

For *odd* M $(= 2m+1)$, rather than giving a particular form of $L, M_1, \ldots, M_{2m}$ that would make

$$\dot{L} = \{L, M_1, \ldots, M_{2m}\} \tag{8}$$

equivalent to the equations of motion

$$\dot{z}_\alpha = -\mathrm{i}\left(\frac{\mathrm{i}}{2}\right)^{n-1} \{z_\alpha, z_{\alpha+1}, \bar{z}_{\alpha+1}, \ldots, z_{\alpha-1}, \bar{z}_{\alpha-1}\} \qquad \alpha = 1, \ldots, n = m+1 \tag{9}$$

let me in this case stress the simple general nature of the construction: think of

$$L = L_1\lambda_1 z_1 + L_2\frac{\bar{z}_1}{\lambda_1} + \cdots + L_{N-1}\lambda_n z_n + L_N\frac{\bar{z}_n}{\lambda_n} \tag{10}$$

and likewise $M_1, \ldots, M_{2m}$, as $N = 2n$-dimensional vectors $\boldsymbol{L}, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_{2m}$ in a vector space V with basis $\lambda_1 z_1, \ldots, \bar{z}_n/\lambda_n$. The desired equivalence of (8) with (9) may then be stated as the requirement that

$$\det\left(\boldsymbol{L}, \boldsymbol{M}_1\boldsymbol{M}_2\ldots\boldsymbol{M}_{2m}\boldsymbol{e}_j\right) = -\mathrm{i}\left(\frac{\mathrm{i}}{2}\right)^{n-1} \hat{\boldsymbol{L}}\cdot\boldsymbol{e}_j \tag{11}$$

where $\boldsymbol{e}_j = (0\cdots010\cdots0)^{\mathrm{tr}}$ and

$$\hat{\boldsymbol{L}} = (L_2, L_1, L_4, L_3, \ldots, L_N, L_{N-1})\,. \tag{12}$$

Multiplying (11) with the jth component of $\boldsymbol{L}$ (or any of the $\boldsymbol{M}$'s), and summing over j, one finds that all $2m+1$ vectors $\boldsymbol{L}, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_{2m}$ have to be perpendicular to $\hat{\boldsymbol{L}}$; in particular

$$\hat{\boldsymbol{L}}\cdot\boldsymbol{L} = 2\,(L_1L_2 + \ldots + L_{N-1}L_N) = 0\,. \tag{13}$$

Choosing $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_{2m}$ to be also perpendicular to $\boldsymbol{L}$, the only remaining condition, obtained by multiplying (11) by $\hat{L}_j$ (and summing), becomes ($\sim$ denoting the projection onto the $2n-2 = 2m$-dimensional orthogonal complement of the $\boldsymbol{L}, \hat{\boldsymbol{L}}$-plane)

$$\det\left(\tilde{\boldsymbol{M}}_1, \ldots, \tilde{\boldsymbol{M}}_{2m}\right) = -\mathrm{i}\left(\frac{\mathrm{i}}{2}\right)^{n-1} \tag{14}$$

which exhibits the large freedom in choosing the $\boldsymbol{M}$'s (for fixed $\boldsymbol{L}$). Similar reasoning applies directly to the real equations (cp [2])

$$\dot{x}_i = \frac{1}{M!}\varepsilon_{ii_1\ldots i_M}\{x_{i_1}, \ldots, x_{i_M}\} \tag{15}$$

the ansatz $L = \sum_{i=1}^{N}\mathbb{L}_i x_i$, $M_1 = \sum \mathbb{M}_{1i}x_i, \ldots$ immediately implies

$$\sum_{i=1}^{N}\mathbb{L}_i^2 = 0 \tag{16}$$

making $L^l, l \in \mathbb{N}$, a harmonic polynomial of $x_1, \ldots, x_N$ (while its integral is time-independent, due to (6) and (8)), irrespective of whether M is odd or even.

References

[1] Hoppe J 1995 On *M*-Algebras, the quantization of Nambu mechanics, and volume preserving diffeomorphisms *Preprint* ETH-TH/95–33

[2] Bordemann M and Hoppe J 1995 Diffeomorphism invariant integrable field theories and hypersurface motions in Riemannian manifolds *Preprint* ETH-TH/95–31, FR-THEP-95-26